



TITLE:

Riemannゼータ関数の近似関数等式に対する平均値公式 (解析数論と数論諸分野の交流)

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Riemann ゼータ関数の近似関数等式 に対する平均値公式

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1 Statement of result.

Let $\zeta(s)$ be the Riemann zeta-function, and define the remainder term $R_1(s)$ in the approximate functional equation for $\zeta(s)$ by

$$R_1(s) = \zeta(s) - \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^s} - \chi(s) \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^{1-s}}$$

where

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{1}{2}\pi s\right) \Gamma(1-s). \quad (1)$$

The aim of this note is to derive the $2k$ -th power moments of the function $|R_1(s)|$ in the critical strip $0 \leq \sigma \leq 1$. We can prove the following

Theorem [1]. For any positive integers k , we have

$$\int_1^T |R_1(s)|^{2k} dt = \begin{cases} \frac{(2\pi)^{k\sigma}}{1-k\sigma} T^{1-k\sigma} + Y_{k,\sigma}(T) & \text{if } 0 \leq \sigma \leq \frac{1}{2k} \\ \frac{(2\pi)^{k\sigma}}{1-k\sigma} T^{1-k\sigma} + D_{k,\sigma} + Y_{k,\sigma}(T) & \text{if } \frac{1}{2k} < \sigma \leq 1 \text{ and } \sigma \neq \frac{1}{k} \\ 2\pi C_k \log T + D_{k,\frac{1}{k}} + Y_{k,\frac{1}{k}}(T) & \text{if } \sigma = \frac{1}{k} \end{cases} \quad (2)$$

with

$$Y_{k,\sigma}(T) = O(T^{\frac{1}{2}-k\sigma}) \quad (5)$$

where the constant $D_{k,\sigma}$ depends on k and σ , and

$$C_k = \int_0^1 \left(\frac{\cos\left(2\pi\left(y^2 - y - \frac{1}{16}\right)\right)}{\cos(2\pi y)} \right)^{2k} dy.$$

Remark. *This theorem includes the fact that*

$$R_1(s) = \begin{cases} \Omega(t^{-\frac{\sigma}{2}}) & \text{if } 0 \leq \sigma < 1, \\ \Omega(t^{-\frac{1}{2}}(\log t)^{\frac{1}{2}}) & \text{if } \sigma = 1. \end{cases}$$

2 Proof of the formula(2).

To prove our theorem, we start with the weak form of the “Riemann-Siegel formula” for $\zeta(s)$: For $0 \leq \sigma \leq 1$, we have

$$\chi(1-s)^{\frac{1}{2}} R_1(s) = (-1)^{[\sqrt{\frac{t}{2\pi}}]-1} \frac{\cos\left(2\pi\left(\delta^2 - \delta - \frac{1}{16}\right)\right)}{\cos(2\pi\delta)} \left(\frac{t}{2\pi}\right)^{-\frac{1}{4}} + O\left(t^{-\frac{3}{4}}\right)$$

where $\delta = \sqrt{\frac{t}{2\pi}} - \left[\sqrt{\frac{t}{2\pi}}\right]$ with $[x]$ being the integer part of x (see [2]). We assume that $T_1 < T_2 \leq 2T_1$. From the above, we have, for $0 \leq \sigma \leq \frac{1}{2k}$,

$$\int_{T_1}^{T_2} |R_1(s)|^{2k} dt = I_1(T_1, T_2) + O\left(\sum_{j=1}^{2k} |I_1(T_1, T_2)|^{1-\frac{j}{2k}} |I_2(T_1, T_2)|^{\frac{j}{2k}}\right), \quad (6)$$

where

$$I_1(T_1, T_2) = \int_{T_1}^{T_2} |\chi(s)|^k \left(\frac{t}{2\pi}\right)^{-\frac{k}{2}} \left(\frac{\cos\left(2\pi\left(\delta^2 - \delta - \frac{1}{16}\right)\right)}{\cos(2\pi\delta)}\right)^{2k} dt$$

and

$$I_2(T_1, T_2) = \int_{T_1}^{T_2} t^{-\frac{3k}{2}} |\chi(s)|^k dt.$$

By using the asymptotic formula of (1), we have

$$|\chi(s)|^k = \left(\frac{t}{2\pi}\right)^{k(\frac{1}{2}-\sigma)} + G_{k,\sigma}(t) \quad (t > 0) \quad (7)$$

where

$$G_{k,\sigma} = O\left(t^{k(\frac{1}{2}-\sigma)-1}\right). \quad (8)$$

It is easily seen that

$$I_2(T_1, T_2) = O\left(T_1^{1-k-k\sigma}\right). \quad (9)$$

From (7) and (8), we get

$$I_1(T_1, T_2) = I_{1,1}(T_1, T_2) + O\left(T_1^{-1} |I_{1,1}(T_1, T_2)|\right), \quad (10)$$

where

$$I_{1,1}(T_1, T_2) = \int_{T_1}^{T_2} \left(\frac{t}{2\pi} \right)^{-k\sigma} \left(\frac{\cos \left(2\pi \left(\delta^2 - \delta - \frac{1}{16} \right) \right)}{\cos(2\pi\delta)} \right)^{2k} dt.$$

Let N_1 be the smallest integer such that $\sqrt{\frac{T_1}{2\pi}} \leq N_1$ and N_2 the largest integer such that $N_2 \leq \sqrt{\frac{T_2}{2\pi}}$. We obtain, for $0 \leq \sigma \leq \frac{1}{2k}$,

$$\begin{aligned} I_{1,1}(T_1, T_2) &= 4\pi \int_0^1 \left(\frac{\cos \left(2\pi \left(y^2 - y - \frac{1}{16} \right) \right)}{\cos(2\pi y)} \right)^{2k} \sum_{n=N_1}^{N_2-1} (y+n)^{1-2k\sigma} dy + O\left(T_1^{\frac{1}{2}-k\sigma}\right) \\ &= \frac{2\pi}{1-k\sigma} \int_0^1 \left(\frac{\cos \left(2\pi \left(y^2 - y - \frac{1}{16} \right) \right)}{\cos(2\pi y)} \right)^{2k} \left\{ \left(y + \sqrt{\frac{T_2}{2\pi}} \right)^{2-2k\sigma} - \left(y + \sqrt{\frac{T_1}{2\pi}} \right)^{2-2k\sigma} \right\} dy \\ &\quad + O\left(T_1^{\frac{1}{2}-k\sigma}\right) \\ &= \frac{(2\pi)^{k\sigma} C_k}{1-k\sigma} (T_2^{1-k\sigma} - T_1^{1-k\sigma}) + O\left(T_1^{\frac{1}{2}-k\sigma}\right). \end{aligned} \quad (11)$$

Substituting (9), (10) and (11) into (6), we obtain the formula (2).

3 Proof of the formulas (3) and (4).

From Lemma 1 in [1], we have, for any positive integers k and $\frac{1}{2k} < \sigma \leq 1$,

$$\int_1^T |R_1(s)|^{2k} dt = J_1(1, T) + J_2(1, T), \quad (12)$$

where

$$J_1(T_1, T_2) = \int_{T_1}^{T_2} |\chi(s)|^k \left| \chi \left(\frac{1}{2k} + it \right) \right|^{-k} \left| R_1 \left(\frac{1}{2k} + it \right) \right|^{2k} dt$$

and

$$J_2(T_1, T_2) = \sum_{j=1}^k \binom{k}{j} \int_{T_1}^{T_2} |\chi(s)|^k \left| \chi \left(\frac{1}{2k} + it \right) \right|^{-(k-j)} \left| R_1 \left(\frac{1}{2k} + it \right) \right|^{2(k-j)} F_k(t)^j dt$$

with

$$F_k(t) = O \left(t^{-\frac{3}{4}} \left| \chi \left(\frac{1}{2k} + it \right) \right|^{-\frac{1}{2}} \left| R_1 \left(\frac{1}{2k} + it \right) \right| + t^{-\frac{3}{2}} \right).$$

Applying (2),(7),(8) and integration by parts, we have, for $k \neq 2$,

$$\begin{aligned} & \int_{T_1}^{T_2} \left| \chi \left(\frac{1}{2k} + it \right) \right|^{-k} \left| R_1 \left(\frac{1}{2k} + it \right) \right|^{2k} dt \\ &= \frac{4\pi}{2-k} C_k \left(\frac{t}{2\pi} \right)^{1-\frac{k}{2}} + \left(\frac{t}{2\pi} \right)^{\frac{1}{2}-\frac{k}{2}} Y_{k, \frac{1}{2k}}(t) \Big|_{T_1}^{T_2} \\ &+ \int_{T_1}^{T_2} H_k(t) \left| R_1 \left(\frac{1}{2k} + it \right) \right|^{2k} dt + \frac{k-1}{2} (2\pi)^{\frac{k-1}{2}} \int_{T_1}^{T_2} t^{-\frac{1+k}{2}} Y_{k, \frac{1}{2k}}(t) dt \end{aligned}$$

and for $k = 2$,

$$\begin{aligned} & \int_{T_1}^{T_2} \left| \chi \left(\frac{1}{4} + it \right) \right|^{-2} \left| R_1 \left(\frac{1}{4} + it \right) \right|^4 dt \\ &= 2\pi C_2 \log t + \sqrt{2\pi} t^{-\frac{1}{2}} Y_{2, \frac{1}{4}}(t) \Big|_{T_1}^{T_2} + \int_{T_1}^{T_2} H_2(t) \left| R_1 \left(\frac{1}{4} + it \right) \right|^4 dt \\ &+ \sqrt{\frac{\pi}{2}} \int_{T_1}^{T_2} t^{-\frac{3}{2}} Y_{2, \frac{1}{4}}(t) dt. \end{aligned}$$

From (5),(7) and (8), we have

$$\begin{aligned} & \int_1^T \left| \chi \left(\frac{1}{2k} + it \right) \right|^{-k} \left| R_1 \left(\frac{1}{2k} + it \right) \right|^{2k} dt \\ &= A_k + B_k(T) + \begin{cases} \frac{4\pi}{2-k} C_k \left(\frac{T}{2\pi} \right)^{1-\frac{k}{2}} & \text{if } k \geq 1 \text{ and } k \neq 2 \\ 2\pi C_2 \log T & \text{if } k = 2 \end{cases} \end{aligned}$$

with

$$B_k(T) = O\left(T^{\frac{1-k}{2}}\right),$$

where the constant A_k depends on k . Hence, integrating by parts, we obtain, for $\frac{1}{2k} < \sigma \leq 1$ and $\sigma \neq \frac{1}{k}$,

$$J_1(T_1, T_2) = O\left(T_1^{1-k\sigma}\right)$$

and

$$J_1(1, T) = \frac{(2\pi)^{k\sigma} C_k}{1-k\sigma} T^{1-k\sigma} + L_{k,\sigma} + O\left(T^{\frac{1}{2}-k\sigma}\right) \quad (13)$$

with a certain constant $L_{k,\sigma}$. Similarly in case $\sigma = \frac{1}{k}$, we have

$$J_1(T_1, T_2) = O(1)$$

and

$$J_1(1, T) = 2\pi C_k \log T + L_{k, \frac{1}{k}} + O\left(T^{-\frac{1}{2}}\right). \quad (14)$$

By using Hölder's inequality and the above, we have, for $\frac{1}{2k} < \sigma \leq 1$,

$$J_2(1, T) = J_2(1, \infty) + O\left(T^{\frac{1}{2} - k\sigma}\right). \quad (15)$$

Substituting (13), (14) and (15) into (12), we obtain the formulas (3) and (4).

References

- [1] I. Kiuchi and N. Yanagisawa, On the mean value formulas for the approximate functional equation of the Riemann zeta-function, preprint.
- [2] C.L.Siegel, Über Riemanns Nachlass zur analytischen Zahlentheorie, Quellen und Studien zur Geschichte der Math. Astr. und Physik, Abt. B, Studien 2(1932), 45-80.